Adiabatic Invariants

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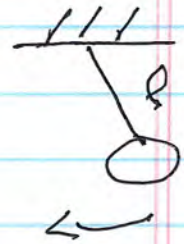
→ 

$\ell = \ell(t)$

o/t $\boxed{\frac{\dot{\ell}}{\ell} < \sqrt{g/\ell}}$ → 2 time scale

→ 'adiabatic' variation of parameter.

→ How describe?

→ Now $\ddot{\theta} + \frac{g}{\ell(t)} \theta = 0$

$\ell(t) = \ell(\underbrace{\epsilon}_{\text{slow}} t)$

$\ddot{\theta} + \frac{g}{\ell(\epsilon t)} \theta = 0$

$\epsilon t = \tau$ $\qquad$ $d/dt =$ [redacted] $\epsilon \frac{d}{d\tau}$

$\Rightarrow dt = \frac{1}{\epsilon} d\tau$

$\ddot{\theta} + \frac{g}{\epsilon^3 \ell(\tau)} \theta = 0$ $\quad \Rightarrow \quad \boxed{\ddot{\theta} + \frac{\omega(\tau)^2}{\epsilon^2} \theta = 0}$

generically points toward WKB.

i.e. generic:

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0$$

$\Rightarrow \tau = \epsilon t$

$$\frac{d^2 x}{d\tau^2} + \frac{\omega^2(\tau)}{\epsilon^2}\, x = 0$$

→ A different look at adiabatic theory ....

One might forego canonical formalism, and simply investigate an oscillator with slowly varying frequency

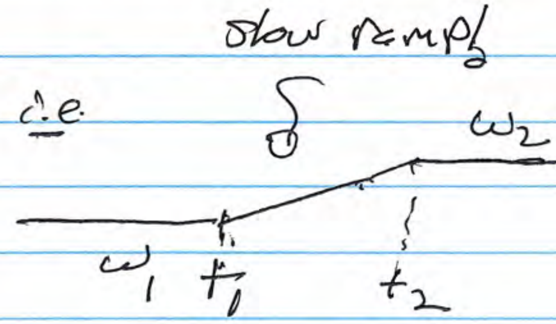
i.e.

$$\ddot{x} + \omega^2 x = 0 \quad \Rightarrow$$

$$\ddot{x} + \omega^2(\epsilon t)\, x = 0$$

slowly varying frequency

i.e.



i.e.

$$\frac{1}{\omega}\frac{d\omega}{dt} \sim O(\epsilon\omega) \ll 1$$

⇒ expect, on basis of previous discussion,

$I \Rightarrow$ adiabatic invariant

i.e. if $a \equiv$ oscillator amplitude, then

$$I = E/\omega = \tfrac{1}{2} m\omega^2 a^2/\omega = m\omega a^2$$

as const

Now, for slowly varying ω, can solve by WKB !

now, $\boxed{\epsilon t = \tau}$

$$\frac{d^2 X}{d\tau^2} + \frac{\omega^2(\tau)}{\epsilon^2} X = 0$$

$$X(\tau) = a_0 e^{i\phi(\tau)/\epsilon}$$

where : $\phi = \phi_0 + \epsilon\phi_1 + \ldots$

eikonal, correction $\Rightarrow$ ampl.

$$\frac{d}{d\tau}\left(a_0 \frac{i\dot{\phi}(\tau)}{\epsilon} e^{i\phi(\tau)} \right) + \frac{\omega(\tau)^2}{\epsilon^2} a_0 e^{i\phi(\tau)} = 0$$

$$\left(-\frac{\dot{\phi}^2}{\epsilon^2} + \frac{i\ddot{\phi}(\tau)}{\epsilon} \right) a_0 e^{i\phi} + \frac{\omega(t)^2}{\epsilon^2} a_0 e^{i\phi(\tau)} = 0$$

need to $O(1/\epsilon)$

$\Rightarrow$

$$\left(-\frac{(\dot{\phi}_0 + \epsilon\dot{\phi}_1)^2}{\epsilon^2} + \frac{i\ddot{\phi}_0(\tau)}{\epsilon} \right) + \frac{\omega(\tau)^2}{\epsilon^2} = 0$$

$$-\frac{\dot{\phi}_0^2}{\epsilon^2} + \frac{\omega(\tau)^2}{\epsilon^2} = 0$$

$$\dot{\phi}_0(T) = \omega(T)$$

$$\phi_0(T) = \int \omega(T)\, dT$$

For next order correction

$$-\frac{2\dot{\phi}_0 \dot{\phi}_1}{\epsilon} + \frac{i\ddot{\phi}_0}{\epsilon} = 0$$

$$\dot{\phi}_1 = i\ddot{\phi}_0 / 2\dot{\phi}_0$$

$$= \frac{i}{2}\frac{d}{dt}\ln\left(\dot{\phi}_0(T)\right)$$

$$\phi_1 = \frac{i}{2}\ln\left(\dot{\phi}_0(T)\right)$$

$$= \frac{i}{2}\ln\left(\omega(T)\right)$$

So

$$x(t) = a_0 e^{i\phi(T)/\epsilon}$$

$$= a_0 e^{i\int \frac{\omega(T)}{\epsilon} dt} e^{i\frac{i}{2}\ln(\omega(T))}$$

$$= a_0 e^{i\int \frac{\omega(T)}{\epsilon} dt} e^{-\frac{\ln}{2}\omega(T)}$$

$$\Rightarrow \quad x(t) = a_0 \, e^{i\int \frac{\omega(t)}{\epsilon} dt} \, e^{-\frac{1}{2}\ln\omega}$$

$$= \frac{a_0}{\sqrt{\omega}} \, e^{i\int \frac{\omega(t)}{\epsilon} \cancel{dt}}$$

$d\tau = \epsilon \, dt$

re-scaling, $t = \tau/\epsilon$; $\omega(\epsilon t)$

$$\boxed{x(t) = \frac{a_0}{\sqrt{\omega}} \, e^{i\int \omega(\epsilon t) \, dt}}$$

WKB soln.

and can observe:

$$\underbrace{\overline{\omega x^2}}_{\text{(cycle) action}} = \omega \, \overline{x^2} = \omega \frac{a_0^2}{\omega} = \text{const}!$$

$\Rightarrow$ Action is invariant, due to frequency modulation of amplitude!

check:
$$I = \frac{1}{2\pi}\oint p\,dq$$

$$= \frac{1}{2\pi}\oint p\,dx$$

$$= \frac{1}{2\pi}\oint m\dot{x}\,dx = \frac{1}{2\pi}\oint m\dot{x}\,\dot{x}\,dt$$

$$I = \frac{1}{2\pi} \oint_{\omega^{-1}} m\dot{x}^2 \, dt$$

$$x(t) = \frac{a_0}{\sqrt{\omega}} \cos(\omega t + \delta)$$

$$\dot{x} = -a_0 \sqrt{\omega} \sin(\omega t + \delta)$$

$\theta = \omega t$ $\qquad$ $\delta \to 0$ $\qquad$ $\omega^2 / (\sqrt{\omega})^2$

$d\theta = \omega dt$

$$I = \frac{1}{2\pi} \oint p \, dq = \frac{1}{2\pi} \oint d\theta \, a_0^2 \, \omega \sin^2\theta \, \frac{d\theta}{\omega}$$

$$= \frac{1}{2} a_0^2 \to \text{real const.!}$$

⇒ the message:

- adiabatic invariants basically a consequence of WKB approximation (time scale sep.!)
- WKB would lead one to adiabatic invariance of action, even if did not realize it.
- need retain WKB correction beyond pure eikonal for freq. modulation of amplitude ⇒ essential.

→ Adiabatic Invariants [and Action - Angle Variables]

i) Adiabatic Invariants

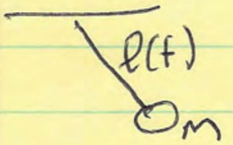
(Bounded phase space ⟺ QPO)

→ Consider finite motion in 1D. Motion characterized by λ parameter, such that:

$$\frac{1}{\lambda}\frac{d\lambda}{dt} \ll \frac{1}{T}$$

↳ period of motion

i.e.



$$\frac{1}{l}\frac{dl}{dt} < \sqrt{g/l}$$

pull on string

thus, $\dot{E}$ will be "small / slow" [i.e. $H = H(\lambda(t), p, q)$

parametric dependence,

Now, $\dfrac{dE}{dt} = \dfrac{\partial H}{\partial t} = \dfrac{\partial H}{\partial \lambda}\dfrac{d\lambda}{dt}$

as λ varies slowly compared to $\omega_0 = 1/T$, can average over t on fast scales, i.e.

$$\frac{d\bar{E}}{dt} = \overline{\frac{\partial H}{\partial \lambda}\frac{d\lambda}{dt}} \cong \overline{\frac{\partial H}{\partial \lambda}}\,\frac{d\lambda}{dt}$$

↳ avg. over motion q, p → fast

{ break avg. on basis time scale separation where $\overline{A} = \frac{1}{T}\int^{\tau} dt\, A(t)$ → holding E, λ

average over fast timescale fixed!

$$\Rightarrow \quad \overline{\frac{\partial H}{\partial \lambda}} = \frac{1}{T}\int_0^{\tau} \frac{\partial H}{\partial \lambda}\, dt$$

Now;

$$\dot{q} = \frac{\partial H}{\partial p}$$

$$dt = \int dq / \partial H/\partial p$$

∴ can take $\int dt \rightarrow \oint \frac{dq}{\partial H/\partial p}$

$\oint$ ↔ complete circuit orbit

so finally,

avg

$$\frac{d\overline{E}}{dt} = \frac{d\lambda}{dt}\left\{\frac{\oint (\partial H/\partial \lambda)\, dq / (\partial H/\partial p)}{\oint dq/(\partial H/\partial p)}\right\}$$

$$\equiv \frac{d\lambda}{dt}\left\langle \frac{\partial H}{\partial \lambda}\right\rangle_{\tau}.$$

Now: - integrations must be performed for fixed, given value of λ (i.e. $\dot{\lambda}/\lambda \ll \omega_0$)

- on such path (n.b. why "path" of interest!), $H = E$ and $p = p(q; E, \lambda)$

$$\therefore \quad H(p, q, \lambda) = E$$

{path for E const.

$$\frac{\partial H}{\partial \lambda} + \frac{\partial H}{\partial p}\frac{\partial p}{d\lambda} = 0$$

$$\Rightarrow \quad \frac{\partial H/\partial \lambda}{\partial H/\partial p} = -\frac{\partial p}{\partial \lambda}$$

plug in previous

$$\therefore \quad \frac{d\bar{E}}{dt} = \frac{d\lambda}{dt}\,\frac{\oint -(\partial p/\partial \lambda)\,dq}{\oint dq\,\partial p/\partial E}$$

$(1/\partial H/\partial p = \partial p/\partial E)$ (fixed λ)

so, re-writing:

$$\frac{d\bar{E}}{dt}\oint dq\,\partial p/\partial E + \frac{d\lambda}{dt}\oint(\partial p/\partial \lambda)\,dq = 0$$

$$\Rightarrow \oint_{\substack{E,\lambda \\ \text{fixed}}} dq \left\{ \frac{\partial p}{\partial E} \frac{d\bar{E}}{dt} + \frac{\partial p}{\partial \lambda} \frac{d\lambda}{dt} \right\} = 0$$

$$\Rightarrow \boxed{\overline{dI/dt} = 0}$$

where $I = \oint_{\substack{E,\lambda \\ \text{fixed}}} \frac{p\, dq}{2\pi}$ $\rightarrow$ { integral taken over path for fixed, given $\underline{E}, \underline{\lambda}$!

$\therefore$ $\rightarrow$ $\boxed{\begin{array}{l} I \text{ const. as } \lambda \text{ varies !} \\ \therefore \; I \text{ adiabatic invariant} \end{array}}$

$\rightarrow$ in general (including higher dimensions)

$$I_c = \oint_\gamma \underline{p} \cdot d\underline{q} = \iint_\sigma d\underline{p} \wedge d\underline{q}$$ { Liouville Thm, again

is Poincares relative integral invariant
(γ closed curve, enclosing σ)

I_c is exact invariant

$$I = \oint p\, dq$$

so $I = I_c \Big|_{E,\lambda \text{ constant}}$

is approximation to Poincare invariant

for $\dot{\lambda}/\lambda < \omega_0$. {Hence adiabatic invariant.

long time scales

Now, adiabatic invariant

$$\underset{\substack{\lambda, E \\ \text{fixed}}}{I = \oint dq\, p/2\pi} \quad \rightarrow \text{ what is it?}$$

so $I = I(E)$

$$= \oint \frac{p\, dq}{2\pi}$$

$$2\pi \frac{\partial I}{\partial E} = \oint \frac{\partial p}{\partial E} dq = \oint \frac{dq}{\partial H/\partial p} = \tau$$

$$\therefore \boxed{\partial I/\partial E = 1/\omega}$$

$$\boxed{\partial E/\partial I = \omega}$$

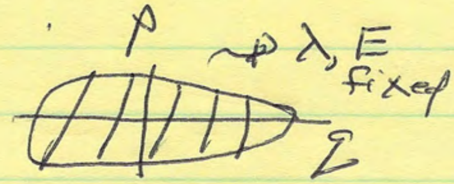
Now, of course:

adiabatic invariant has geometrical significance.

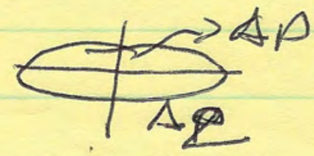
$$I = \oint_{E,\lambda} \frac{p\,dq}{2\pi} = \iint_{E,\lambda} \frac{dp\,dq}{2\pi}$$

$\therefore$ I corresponds to enclosed area!

i.e.



e.g. $$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 = E$$



$$\Delta p = (2mE)^{1/2}$$
$$\Delta q = (2E/m\omega^2)^{1/2}$$

$$\text{Area} = \pi\,\Delta q\,\Delta p = 2\pi E/\omega$$

$$\boxed{I = E/\omega}$$

$\Rightarrow$ for oscillator adiabatic invariant is <u>action</u>, E/ω.

$\therefore$ $\dot{\ell}/\ell < \omega_0$ $\Rightarrow$

$E \sim \omega \sim \sqrt{g/\ell}$

Physics 200 A, B — Mechanics

→ Adiabatic Invariants: Review

→ if $H = H(p, q, \lambda(t))$

$\lambda(t)$: parametric dependence

with a) periodic motion, for fixed λ.

b.) $\frac{1}{\lambda}\frac{d\lambda}{dt} < \omega$

$\frac{1}{\lambda}\frac{d\lambda}{dt}$: rate of change of parameter

ω: motion frequency

then $I_\lambda = \oint_{C_\lambda} p \cdot dq$ ≡ action computed at fixed value of λ is adiabatic invariant

→ adiabatic invariant is C.O.M. on time scales $\tau > \omega^{-1}$.

→ adiabatic invariant is intrinsically / implicitly referenced to a given time scale. Same system can manifest multiple adiabatic invariants on different time scales.

→ $I = \oint_C p \cdot dq$ → Poincare - Cartan Invariant → exact C O M

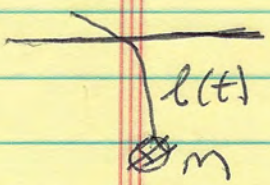
to calculate explicitly, need integrable motion (as in explicit representation of action-angle var.)

but

— $I_\lambda = \oint_{C_\lambda} \underline{p}\cdot d\underline{q}$ is <u>approximation</u> to I,

computed for <u>fixed λ</u>. $\dot{I}_\lambda \cong 0$ for $t \gg \omega^{-1}$.

<u>Examples</u>: i) <u>Pendulum — the Prototype</u>



$$\frac{\dot{\ell}(t)}{\ell} < \sqrt{g/\ell}.$$

How does Θ vary with ℓ?

$I = E/\omega$, understood $E = \overline{E}$

$\omega = \sqrt{g/\ell}$

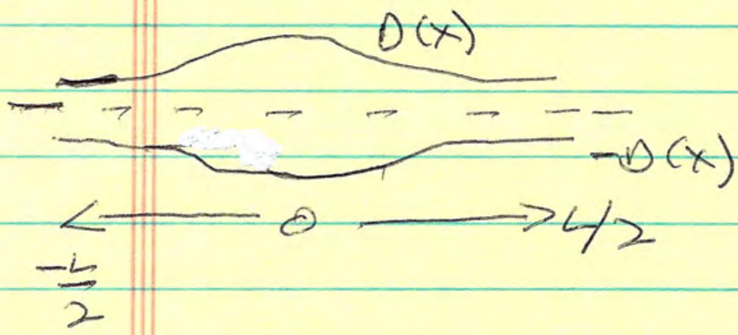
$$\overline{E} = \frac{m}{2}\ell^2\overline{\dot{\Theta}^2} + mg\ell\frac{\overline{\Theta^2}}{2}$$

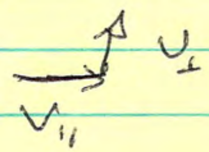
$$= mg\ell\overline{\Theta^2}$$

$$I = m\sqrt{g}\,\ell^{3/2}\,\overline{\Theta^2}$$

so $\Theta_{rms} \sim \ell^{-3/4}$ i.e. amplitude decreases as length increases more generally, $\dfrac{\theta_{rms}(t)}{\theta_{rms}(0)} \sim \left(\dfrac{\ell(0)}{\ell(t)}\right)^{3/4}$

2.) Mechanical Mirror





$\tau_{b_\perp} \sim (V_\perp/2D)^{-1} \rightarrow$ ⊥ bounce time

$\tau_b \ll L/V_\parallel$

many bounces(⊥) in time to sense curvature of D.

now,

$$2\pi I = \int_{-D}^{D} m v_\perp \, dy + \int_{D}^{-D} (-m v_\perp) \, dy$$

$$= 4 m D V_\perp$$

$$\boxed{I = \frac{2}{\pi} D m V_\perp}$$

Adiabatic Invariant

$$E = \frac{1}{2} m \left(V_\perp^2 + V_\parallel^2\right)$$

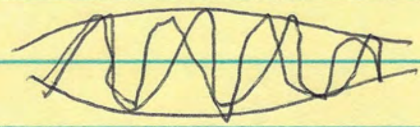
$$\therefore \quad E = \frac{1}{2} m \left(\frac{\pi^2}{4D^2} \frac{I^2}{m^2} + V_\parallel^2 \right)$$

$$\Rightarrow \quad \frac{2E}{m} = \frac{\pi^2}{4D^2(x)} \frac{I^2}{m^2} + V_\parallel^2$$

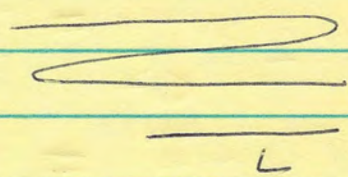
$$\boxed{V_\parallel^2 = \frac{2E}{m} - \frac{\pi^2}{4D^2(x)} \frac{I^2}{m^2}}$$

<u>if</u> $\quad \frac{\pi^2 I^2}{4D(x)^2 m^2} > \frac{2E}{m} \quad \Rightarrow$ particle reflected in throat of mirror

<u>i.e.</u>



i.e. on avg.



$$I = \frac{2}{\pi} D(x_0) \, \underbrace{m V_{\perp 0}}_{\text{injection momentum}}$$

$D(x_0)$: separation at injection

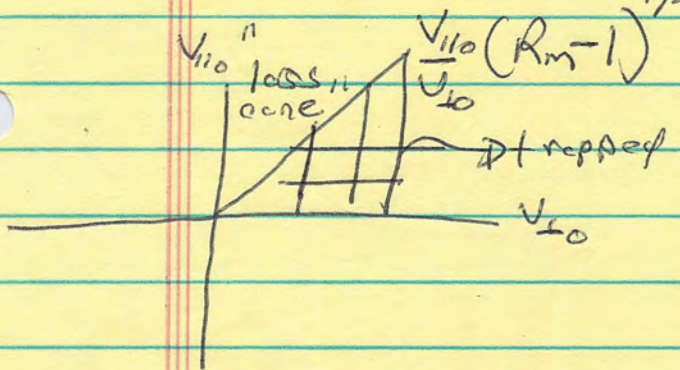
$$\Rightarrow \quad \left(\frac{D(x_0)}{D(x)} \right)^2 V_{\perp 0}^2 > 2E/m$$

$x < L$

∴ for mirroring / trapping:

$$\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} < \underbrace{\left(\frac{D(x_0)}{D(x)}\right)^2}_{\text{"mirror ratio"}} - 1$$

i.e. trapping condition: $\left\{ \boxed{\frac{v_{\parallel 0}^2}{v_{\perp 0}^2} < \overbrace{\left(\frac{D(x_0)}{D(x)}\right)^2}^{R_m} - 1} \right.$



n.b. { some class of particles always lost

Now, for trapped / mirroring particles, can determine reflection point x_R from:

$$v_\parallel^2 = \frac{2E}{m} - \frac{\pi^2}{4D(x_R)^2}\frac{I^2}{m^2} = 0$$

$x_R \leq L/2$

then : for $t > \tau_{b\parallel} = \oint \frac{dx}{|v_\parallel|}$

$\underbrace{\tau_{b\parallel}}$ parallel bounce time for trapped

can have "2^{nd}" adiabatic invariant on time scale $\tau_{b\parallel} >> \tau_{b\perp}$

$$J_\parallel = \oint dx \, p_\parallel$$

$J_\parallel \Rightarrow$ "2^{nd}" ad. inv. — $\parallel$ bounce inv.

$$\frac{p_\parallel^2}{2m} = E - \frac{\pi^2 I^2}{8 p(x)^2}/m$$

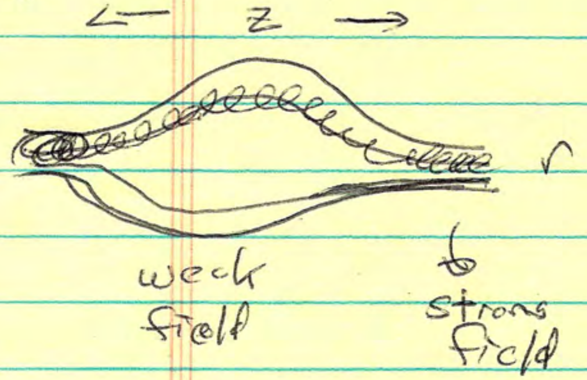
$J_\perp \Rightarrow 1^{st}$ ad inv. — $\perp$ bounce inv.

$$\tau_\perp < \tau < \tau_\parallel < \tau'$$

$\tau_\perp$: $\perp$ bounce time; τ: time of $J_\perp$ ad. inv.; $\tau_\parallel$: $\parallel$ bounce time; τ': time of $J_\parallel$ ad. inv.

N.B.: Can expect 1 adiabatic invariant per closed cyclic orbit (n.b. cyclic orbit a/a' action variable sense).

3.) Magnetic Mirror – basis for mechanical mirror



$$\underline{\nabla} \cdot \underline{B} = 0$$

$$\frac{\partial}{\partial z} B_z + \nabla_r B_r = 0$$
$$\neq 0$$

Now, consider rate of change of ⊥ Energy

$$\frac{d}{dt}\left(\frac{m V_\perp^2}{2}\right) = q\, \underline{E}_\perp \cdot \underline{V}_\perp$$

avg over 1 cyclotron orbit ⟹

$$\delta\left(\frac{m V^2}{2}\right) = \int_{\Omega^{-1}} dt\, q\, \underline{E}_\perp \cdot \underline{V}_\perp \qquad \underline{v}\, dt = \ell$$

change in energy in 1 cyclotron orbit

$$= \int_{\substack{\rho \\ \text{gyro circle}}} d\underline{\ell} \cdot \underline{E}_\perp q = q \int \underline{E} \cdot d\underline{\ell}$$

$\rho \to$ gyro-radius

$$= \int d\underline{a}\, q \cdot \underline{\nabla} \times \underline{E}$$

$$= -\int_{\text{gyrodisk}} d\underline{a} \cdot \left(\frac{q}{c}\frac{\partial \underline{B}}{\partial t}\right)$$

$$\cong -\pi \rho^2 \, \frac{q}{c}\frac{\partial B}{\partial t}$$

$\rho^2 = V_\perp^2/\Omega^2$

$\Rightarrow$

$$\delta\left(\frac{mV_\perp^2}{2}\right) \cong -\pi \frac{q}{c} \frac{V_\perp^2}{\frac{q^2 B^2}{m^2 c^2}} \frac{\partial B}{\partial t}$$

$$= -\frac{mV_\perp^2}{\Omega} \frac{\pi}{B} \frac{\partial B}{\partial t}$$

but $\underbrace{\delta B}_{\substack{\text{change in} \\ \text{1 cyclotron} \\ \text{period}}} = \underbrace{\frac{2\pi}{\Omega}}_{\tau_c} \frac{\partial B}{\partial t}$

$\therefore$

$$\delta\left(\frac{mV_\perp^2}{2}\right) = -\frac{mV_\perp^2}{2} \frac{1}{B} \delta B$$

$\Rightarrow$ $\boxed{\delta\left(\frac{mV_\perp^2}{2B}\right) = 0}$ $\Rightarrow$ adiabatic time variation in $B \Rightarrow$ heating

so $\boxed{\mu = mV_\perp^2/2B}$ $\rightarrow$ magnetic moment adiabatic invariant on $t \gg \Omega^{-1}$

Now, for mirroring:

$$\frac{1}{\cancel{2}}\cancel{m}(V_\parallel^2 + V_\perp^2) = \frac{1}{\cancel{2}}\cancel{m}(V_{\parallel 0}^2 + V_{\perp 0}^2)$$

$$\frac{\cancel{m}\, V_\perp^2(0)}{\cancel{2}B(0)} = \frac{\cancel{m}\, V_\perp^2(\ell)}{\cancel{2}B(\ell)}$$

$$V_\parallel^2(0) + V_\perp^2(0) = V_\parallel^2 + \frac{B(\ell)}{B(0)} V_\perp^2(0)$$

$$V_\perp^2(0)\left(1 - \frac{B(\ell)}{B(0)}\right) = V_\parallel^2(\ell) - V_\parallel^2(0)$$

for confinement: $V_\parallel^2(\ell) = 0 \Rightarrow$

so $$\boxed{\frac{V_\parallel^2(0)}{V_\perp^2(0)} < \underbrace{\frac{B(\ell)}{B(0)}}_{\text{mirror ratio}} - 1}$$

obvious analogy to:

$$\frac{V_{\parallel 0}^2}{V_{\perp 0}^2} < \frac{D(x_0)}{D(x_R)} - 1$$

$D(\ell) \leftrightarrow 1/D(x_R) \rightarrow$ strong $B \rightarrow$ frequent gyration, frequent bouncing

$B(0) \leftrightarrow 1/D(x_0) \rightarrow$ weak $B \rightarrow$ less frequent bouncing, gyration.

Similarly, can define bounce invariant:

$$J_{\parallel} = \oint d\ell \left[2m\left(E - \mu B(\ell)\right)\right]^{1/2}$$

longitudinal action

i.e. $V_{\parallel}^2(\ell) = V_{\parallel}^2(0) + V_{\perp}^2(0) - \mu B(\ell)$

etc.

squeeze → energy gain

N.B.: Treatment of adiabatic invariants given here corresponds to lowest order p.t. in $\frac{1}{\lambda}\frac{d\lambda}{dt}\Big/\omega < 1$

$$\underbrace{}_{\text{"}O(\epsilon)\text{" here.}}$$